Arithmetic Sequences and Series

Worksheet Solutions

1. Express each of the following series in sigma notation.

   (a) $1 + 2 + 3 + ... + n$

   (b) $1 + \frac{1}{4} + \frac{1}{9} + ... + \frac{1}{n^2}$

   (c) $\frac{1}{2} + \frac{4}{5} + \frac{3}{4} + ... + \frac{2n+2}{n^2+4}$

   (d) $-1 + 1 - 2 + ... + (-1)^n(n-1)! + ...$

   (e) $\cos(\theta) + \cos^2(\theta) + \cos^3(\theta) + ... + \cos^{88}(\theta)$

a) $\sum_{r=1}^{n} r$

b) $\sum_{r=1}^{n} \frac{1}{r^2}$

c) $\sum_{r=0}^{n} \frac{2r+2}{r^2+4}$

d) $\sum_{r=1}^{\infty} (-1)^r (r-1)!$

e) $\sum_{r=1}^{88} (\cos(\theta))^r$

2. For each of the following arithmetic sequences:
   i. Find the 64th term of the sequence.
   ii. Find the sum of the first 14 terms of the sequence.

   (a) $-8, -4, 0, 4, \ldots$

   (b) $62, 61, 60, 59, \ldots$

   (c) $7, 23, 39, 55, \ldots$

   (d) $1, \frac{3}{2}, 2, \frac{5}{2}, \ldots$

   (e) $-1, -7, -13, -19, \ldots$

$$S = \frac{n}{2}\left(2a + (n-1)d\right)$$

a) $a = -8, \quad d = 4, \quad a_k = -8 + 4(k-1)$

$a_{64} = -8 + 4(63) = 244$

$S_{14} = \frac{14}{2}\left(-16 + 13(4)\right) = 252$

b) $a = 62, \quad d = -1, \quad a_k = 62 - (k-1)$

$a_{64} = 62 - 63 = -1$

$S_{14} = \frac{14}{2}\left(2(62) + 13(-1)\right) = 777$

c) $a = 7, \quad d = 16, \quad a_k = 7 + 16(k-1)$

$a_{64} = 7 + 16(63) = 1015$

$S_{14} = 7\left(14 + 13(16)\right) = 1554$

d) $1, \frac{3}{2}, 2, \frac{5}{2}, \ldots$

$a = 1$, $d = \frac{1}{2}$, $a_k = 1 + \left(\frac{1}{2}\right)(k-1)$

$a_{64} = 1 + \frac{1}{2}(63) = \frac{65}{2}$

$S_{14} = 7\left(2 + \frac{1}{2}(13)\right) = \frac{119}{2}$

e) $-1, -7, -13, -19, \ldots$

$a = -1$, $d = -6$, $a_k = -1 + -6(k-1)$

$a_{64} = -1 + -6(63) = -379$

$S_{14} = 7\left(-2 + (-6)(13)\right) = -560.$

3. An arithmetic sequence has first term 2 and final term 48.

   Given that the sequence has twelve terms, find the sum of all terms in the sequence.

$a = 2, \quad l = 48$

$48 = 2 + (12-1)d$

$\Rightarrow \quad d = \frac{46}{11}$

$S_{12} = \frac{12}{2}\left(4 + 11\left(\frac{46}{11}\right)\right)$

$= 300.$

4. An arithmetic sequence has 11th term 30. The sum of the first 11 terms of this sequence is 55. What is the common difference for this sequence?

$a_{11} = 30, \quad S_{11} = 55$

$30 = a + 10d, \quad 55 = \frac{11}{2}(2a + 10d)$

$30 = a + 10d, \quad 110 = 22a + 110d$

$$\begin{aligned} 110 &= 22a + 110d \\ - \quad 330 &= 11a + 110d \end{aligned}$$

$-220 = 11a \Rightarrow a = -20$

$10d = 30 - a = 50$

$d = 5$

5. An arithmetic sequence has final term $l = 6$ and common difference $d = 3$.

   Given that the sum of all terms in the sequence is 0, find the first term of the sequence and the total number of terms in the sequence.

$l = 6, \quad d = 3$

$S_n = \frac{n}{2}\left(2a + 3(n-1)\right) = 24$

$6 = a + (n-1)3$,

$0 = n\left(2a + 3(n-1)\right)$

$6 = a + 3(n-1) \Rightarrow a = 6 - 3(n-1)$

$= 6 - 3n + 3$

$a + 3n = 9$

$0 = n\left(2(9-3n) + 3(n-1)\right)$

$\Rightarrow$ $0 = n(18 - 6n + 3n - 3)$

$0 = n(15 - 3n)$

$0 = 15n - 3n^2 \Rightarrow 3n^2 - 15n = 0$

$3n(n-5) = 0 \Rightarrow n = 0$ or $n = 5$

$n = 0$ doesnt make sense

$n = 5, \; a = -6$.

6. The sum of the first 32 terms of an arithmetic sequence is 1864. The sum of the first 10 terms of the same sequence is 395.

   (a) Find an expression for the nth term of this sequence.

   (b) Find the 100th term of the sequence.

   (c) After how many terms does the sum of the sequence exceed 2080?

a) $S_{32} = 1864 = \frac{32}{2}(2a + 31(d))$

$\Rightarrow 1864 = 16(2a + 31d)$

$\Rightarrow 1864 = 32a + 496d$

$S_{10} = 395 = \frac{10}{2}(2a + 9d)$

$\Rightarrow 395 = 10a + 45d$

$\Rightarrow 1264 = 32a + 144d$

$600 = 352d \Rightarrow d = \frac{75}{44}$

$a = \frac{2801}{88}$

$a_k = \frac{2801}{88} + \frac{75}{44}(k-1)$

$$a_{100} = \frac{2801}{88} + 99\left(\frac{75}{44}\right) = \frac{17651}{88}$$

c) $$\frac{n}{2}\left(2\left(\frac{2801}{88}\right) + (n-1)\left(\frac{75}{44}\right)\right) > 2080$$

$$n\left(\frac{2801}{44} + (n-1)\left(\frac{75}{44}\right)\right) > 4160$$

$$2801n + 75n(n-1) > 183040$$

$$2801n + 75n^2 - 75n > 183040$$

$$75n^2 + 2726n - 183040 > 0$$

$$n > \frac{1}{75}\left(\sqrt{15585769} - 1363\right)$$

$$n > 34.4$$

$$n = 35.$$

7. A container is leaking at a constant rate of $\frac{3}{7}$ml per minute. Given that the container is initially full of 6.4 litres of water, calculate how long it will take for all water to have left the container.

$n = n-1$ mins

$a_k = 6400 - (k-1)$

$0 = 6400 - \frac{1}{7}(k-1)$

$0 = 44800 - (k-1)$

$k - 1 = 44800$

$k = 44801 = 44,800$ mins

8. Consider an arithmetic sequence with first term a and common difference d where $a, d \in \mathbb{R}$.

   Let $S = a + (a + d) + (a + 2d) + ... + (a + (n - 1)d$.

   Show that $S = \frac{n}{2}(2a + (n - 1)d)$.

$$S = (a + (n-1)d) + (a+(n-2)d) + \dots + (a+d) + a$$

$$S = a + (a+d) + \dots + (a+(n-2)d) + (a+(n-1)d)$$

$$2S = 2a+(n-1)d + 2a+(n-1)d + \dots + 2a+(n-1)d + 2a+(n-1)d$$

$$= n(2a+(n-1)d)$$

$$S = \frac{1}{2}n(2a+(n-1)d).$$